

A Discussion on the Teleportation Protocol for States of N Qubits.

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Abstract

In this paper, we want to present a simple and comprehensive method to implement teleportation of a system of N qubits and its discussion on the corresponding quantum circuit. The paper can be read for nonspecialists in quantum information.

I. MOTIVATION

Quantum Information and quantum computation are subjects that are recently received an enormous interest in the scientific community. Textbooks of a high pedagogical value have been written on these subjects. Among them, we can quote the monographs from Nielsen and Chuang¹ and Preskill². Nevertheless, some important procedures, which in addition have not been treated in the monographs, need a reformulation and a presentation that make them accessible to the physics teacher.

We have chosen the teleportation of N qubits because i.) this is a relevant subject. In fact, once we have an algorithm that can teleport a quantum state, we immediate ask for a natural and simple generalization that can enable us to transmit a large amount of information at long distance. ii.) protocols to teleport N qubit states have already been published⁵. We want to introduce here another N qubit teleportation protocol based in the previous ones⁵, but written in a simpler new fashion that pretends to be more useful and more clear to the average physicists interested in these subjects.

The possibility of teleportation of a qubit state has been suggested in 1993 by Bennet et al³. Later, in 1998, Brassard developed a quantum circuit in order to implement one qubit teleportation⁴. In the description of the teleportation of an N qubit state, we shall also introduce a circuit that will do the job. The advantage of the circuit notation is that it makes easier the comprehension of the process through a visualization of it.

The protocol for the teleportation of one qubit state is very well known and has been discussed in textbooks as for instance in monograph from Nielsen and Chuang¹. The best known teleportation protocol can be summarized in few words as follows: Let us start with the qubit that we want to teleport, represented by the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|0\rangle$ denotes⁷ “spin up” and $|1\rangle$ “spin down”, and let us consider the auxiliar two qubit state represented by the Bell state:

$$|\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \quad (1)$$

Let us consider the following three qubit state:

$$|\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\{\alpha|0\rangle[|00\rangle + |11\rangle] + \beta|1\rangle[|00\rangle + |11\rangle]\}. \quad (2)$$

The first qubit correspond to the state that we want to teleport. This and the second qubit are assumed to be and to remain in the hands of Alice, the sender. The third qubit is brought by Bob, the receiver, to his location. Alice can thus manipulate her two qubits and she does it as follows: First, she makes them to pass through a CNOT gate. The CNOT gate flips the second qubit if the first one is in the state $|1\rangle$ and keeps it unchanged if the former is $|0\rangle$.

Then, Alice makes her first qubit passing through a Hadamard gate. We recall that a Hadamard gate produces the following changes:

$$|0\rangle \mapsto \frac{1}{2}\{|0\rangle + |1\rangle\} \quad ; \quad |1\rangle \mapsto \frac{1}{\sqrt{2}}\{|0\rangle - |1\rangle\}. \quad (3)$$

Thus, the resulting three qubits state can be written as follows¹:

$$\begin{aligned} |\psi_I\rangle = \frac{1}{2}\{ &|00\rangle(|\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) \\ &+ |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle) \} \end{aligned} \quad (4)$$

Then, Alice produces a measurement on her two qubits. She can have one out of four results only: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. By means of an open classical system of communication (telephone, e-mail, etc), Alice communicates to Bob the result obtained. Accordingly, Bob produces an operation on his qubit so as to obtain the original state (Since this operation is performed in an environment different from the original lab where the first qubit was produced, we can say that the original state has been teleported). This operation is:

Alice result	Bob's operation on his qubit
$ 00\rangle$	Does nothing
$ 01\rangle$	X gate
$ 10\rangle$	Z gate
$ 11\rangle$	ZX gate

We recall that the X and Z gate are the σ_x and σ_z Pauli matrices respectively⁸. The ZX gate means that we first apply the X gate and then the Z gate, this notation is copied from the usual algebraic manipulation according to which the first operation lies in the right the second on its left and so on. All these operations can be written as $Z^{M_1}X^{M_2}$, where M_i , with $i = 1, 2$, are either 0 or 1.

Thus, if for example the measurement of the first two qubits give $|00\rangle$, the third qubit must be already in the state we want teleport as shown in (4). Bob does not need to do anything as he has the wanted state. This operation *do nothing* is written in algebraic form as Z^0X^0 . If Alice gets $|01\rangle$, then Bob applies $X \equiv Z^0X^1$ to his qubit, etc.

Note that the gate X is equivalent to the NOT gate that produces the flips $|0\rangle \longleftrightarrow |1\rangle$ and $|1\rangle \longleftrightarrow |0\rangle$, while Z just changes the sign in front of $|1\rangle^1$.

It is noteworthy to recall the obvious fact that the subject of quantum teleportation are qubit states and not any kind of particles with or without mass (qubit could be implemented using photon polarization).

The remainder of this paper is organized as follows: In the brief Section II, we introduce a useful notation that helps in abbreviating lengthly formulas. It is in the long section III where we discuss the teleportation algorithm. The algorithm is obtained by induction on the number of qubits of the state to be teleported. This induction procedure is discussed in detail in subsection III B. Previously, in subsection III A, we have derived an important intermediate formula. Finally, on subsection III C, we present the quantum circuit for the teleportation with careful explanation of all its constituents.

The paper is written in a style that can be read for physicists, quantum chemists and computer scientists without previous experience in quantum information theory, thus being intended for a wide audience.

II. USEFUL NOTATIONS.

As we intend a thoroughly discussion on a rather cumbersome manipulation as teleportation of N qubits is and, at the same time, we pretend to give an accessible version of this operation, it seems natural to search for a notation that can help us in our goal. To this end, we choose the following:

- i.) Let us denote by k_n a chain of n bits, where k is a natural number and k_n is the chain that represents k in terms of these n bits.
- ii.) One of the tools available is the so called Iverson delta. In order to define it, we first

associate to each property p a number, as introduced by Iverson⁶, such that

$$[p] := \begin{cases} 1 & \text{if } p \\ 0 & \text{if } \sim p \end{cases} . \quad (5)$$

This is called the Iverson notation⁶. After (5), we define the Iverson delta as follows: let i and k two natural numbers with chain of digits k_n and i_n . Then,

$$\ddot{\delta}_{i,k} := [(i_n \text{ AND } k_n) \text{ have an odd number of bits with the digit 1}] . \quad (6)$$

We are listing below some interesting properties of the Iverson delta. Their proof is not essential in our presentation (and otherwise easy to obtain) and we omit it here:

$$1.- (-1)^{\ddot{\delta}_{2i+1,2k+1}} = (-1)^{\ddot{\delta}_{i,k}+1} .$$

$$2.- (-1)^{\ddot{\delta}_{2i+1,2k}} = (-1)^{\ddot{\delta}_{i,k}} .$$

$$3.- (-1)^{\ddot{\delta}_{2i,2k+1}} = (-1)^{\ddot{\delta}_{i,k}} .$$

$$4.- (-1)^{\ddot{\delta}_{2i,2k}} = (-1)^{\ddot{\delta}_{i,k}} .$$

iii.) This kind of replacement is very usual:

$$\sum_{a_1 \dots a_n=k}^l \equiv \sum_{a_1=k}^l \dots \sum_{a_n=k}^l . \quad (7)$$

So far the explanation of notation to be used in the sequel. In the next section, we start with our presentation.

III. TELEPORTATION ALGORITHM.

We want to teleport an arbitrary pure state of N qubits that we shall denote by $|\psi_N\rangle$. Then, $|\psi_N\rangle$ is a vector state of the tensor product of N times the two dimensional Hilbert space \mathbb{C}^2 , where qubits dwell⁹.

Once the state $|\psi_N\rangle$ has been prepared, we need a device to teleport it. In the case of $N = 1$, we have seen that 2 additional or auxiliary (also called ancillary) qubits are needed. In our case, we can expect that we shall require $2N$ auxiliary qubits. Then, we have $3N$ qubits that we shall distribute into three groups.

First of all the N qubits whose state we want to teleport. These qubits will make the first group and its state denoted as $|\psi_N\rangle_1$, where we have added the subindex 1 accordingly. The state $|\psi_N\rangle_1$ can be written in terms of the N qubit basis $|i_N\rangle$ as

$$|\psi_N\rangle_1 = \sum_{i=0}^{2^N-1} \alpha_i |i_N\rangle_1.$$

We assume that $|\psi_N\rangle_1$ is normalized.

The second group will be formed by the first N auxiliary qubits. Before the beginning of the teleportation procedure all them are prepared to be in the state $|0\rangle$. Then, the quantum state for the system of these N qubits is $|00\dots 0\rangle_2$ with N zeroes. If we use the notation described in the previous section, a chain of N zeros is described by 0_N , so that the state of this second group of qubits is here denoted as $|0_N\rangle_2$.

By the same arguments, we write the initial state of the third group of qubits as $|0_N\rangle_3$.

The teleportation protocol for a N qubit state can be looked as a generalization of the $N = 1$ case. With this idea in mind, let us take the first N auxiliar qubits in the collective state $|0_N\rangle_2$ and let them pass through respective Hadamard gates.

Once this operation has been completed, take the auxiliary qubits of the third group and make the following CNOT operations; the $(N + 1)$ -th auxiliary qubit (after Hadamard!) with the $(2N + 1)$ -th, the $(N + 2)$ -th with the $(2N + 2)$ -th and so on. See Figure 1.

The final result is the complete entanglement of the $2N$ auxiliary qubits. The collective state of the system of all these qubits resulting after these manipulations is a generalization of the Bell state (1) used in the $N = 1$ case. This generalized Bell state has the following form:

$$\frac{1}{\sqrt{2^N}} \sum_{j=0}^{2^N-1} |j_N j_N\rangle_{23}. \quad (8)$$

For instance, for $N = 2$, this sum gives:

$$\frac{1}{2} \{ |0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle \}. \quad (9)$$

At this point, Alice (sender) and Bob (receiver) move away from each other. The generalization of the $N = 1$ case suggests that Alice keeps the $2N$ first qubits, i.e., the N qubit state to be teleported $|\psi_N\rangle_1$ and the first N auxiliary qubits in their final state $|\rangle_2$.

Next, Alice performs a CNOT operation between the qubit k and the qubit $k + N$ for all $k = 1, 2, \dots, N$ (see Figure 1). This produces the following state of the $3N$ qubit system:

$$\frac{1}{\sqrt{2^N}} \sum_{i=0}^{2^N-1} \alpha_i |i_N\rangle_1 \sum_{j=0}^{2^N-1} |(j \text{ XOR } i)_N j_N\rangle_{23}. \quad (10)$$

We recall that the operation¹⁰ $j \text{ XOR } i$ means $j+i$ modulus 2, i.e., $0+0=0$, $0+1=1+0=1$ and $1+1=0$. For example, if $N=2$ and $j=2$ ($j_2=10$) and $i=3$ ($i_2=11$), we have that $(j \text{ XOR } i)_N = 01 \equiv 1$.

The next operation performed by Alice is applying a Hadamard gate to each of the first N qubits (which original state we want to teleport, see Figure 1). In subsection III A, we show that the final state of the $3N$ qubit system is given by

$$\frac{1}{2^N} \sum_{i,j,k=0}^{2^N-1} |k_N(j \text{ XOR } i)_N\rangle_{12} (-1)^{\delta_{i,k}} \alpha_i |j_N\rangle_3. \quad (11)$$

Nevertheless, (11) is not the most useful form of the state of this entanglement of $3N$ qubits. In subsection III B, we shall show that formula (11) is equal to

$$\frac{1}{2^N} \sum_{a_1 \dots a_N=0}^1 |a_1 \dots a_{2N}\rangle_{12} \left(\bigotimes_{k=a_{N+1}}^{a_{2N}} X^k \right) \left(\bigotimes_{l=a_1}^{a_N} Z^l \right) |\psi_N\rangle_3. \quad (12)$$

Then, Alice makes a measurement of her $2N$ qubits. Assume that the result is $|a_1 \dots a_{2N}\rangle_{12}$. Then, the state of the N qubits own by Bob is given by

$$|\varphi\rangle_3 = \left(\bigotimes_{k=a_{N+1}}^{a_{2N}} X^k \right) \left(\bigotimes_{l=a_1}^{a_N} Z^l \right) |\psi_N\rangle_3. \quad (13)$$

In order to obtain the original state $|\psi_N\rangle$, Bob must multiply $|\varphi\rangle_3$ in (13) by the inverse of the operator $\left(\bigotimes_{k=a_{N+1}}^{a_{2N}} X^k \right) \left(\bigotimes_{l=a_1}^{a_N} Z^l \right)$. Hence, teleportation of $|\psi_N\rangle$ is completed.

Although formulas (11) and (12) describe the same $3N$ qubit entangled state, we see that (11) is useless for teleportation while the usefulness of (12) is quite obvious. However, the derivation of (12) from (11) is not immediate and needs some discussion. This is presented in subsection III B.

We recall that the whole procedure is described by a circuit. This is presented in Figure 1 and subsection III C.

A. Proof of (11).

Our next goal is to show that (10) plus the operation of passing the first N qubits through respective Hadamard gates gives (11). First of all, let us consider the state $|i_N\rangle$ of a system of N qubits. Each qubit, will pass through a Hadamard gate, this action is represented as $H^{\otimes N}$, where H stands for Hadamard gate. Then we have to show that

$$H^{\otimes N}|i_N\rangle = \frac{1}{\sqrt{2^N}} \sum_{k=0}^{2^N-1} (-1)^{\ddot{\delta}_{i,k}} |k_N\rangle. \quad (14)$$

We shall prove this result by induction on N . For $N = 2$, we call x and y to the first and second qubit. We use the properties of the Iverson delta. Then,

$$\begin{aligned} H^{\otimes 2}|xy\rangle &= H|x\rangle H|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) \frac{1}{\sqrt{2}}(|0\rangle + (-1)^y|1\rangle) \\ &= \frac{1}{2}[|00\rangle + (-1)^y|01\rangle + (-1)^x|10\rangle + (-1)^{x+y}|11\rangle] \\ &= \frac{1}{2} \sum_{k=0}^3 (-1)^{\ddot{\delta}_{xy,k}} |k_2\rangle, \end{aligned} \quad (15)$$

which proves (14) for $N = 2$.

Now, we assume that the result is true for $N = 3, 4, \dots, n$. Under this hypothesis, if we prove it for $N = n + 1$ it would be shown for any value of N by induction. We start with the $n + 1$ qubit state $|i_{n+1}\rangle \equiv |j_n\rangle |x\rangle$ and make each qubit pass through a Hadamard gate:

$$\begin{aligned} H^{\otimes n+1}|i_{n+1}\rangle &= H^{\otimes n+1}|j_n\rangle |x\rangle = H^{\otimes n}|j_n\rangle H|x\rangle \\ &= \left[\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} (-1)^{\ddot{\delta}_{j,k}} |k_n\rangle \right] \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x|1\rangle) \\ &= \frac{1}{\sqrt{2^{n+1}}} \sum_{k=0}^{2^n-1} \left((-1)^{\ddot{\delta}_{j,k}} |k_n\rangle |0\rangle + (-1)^{\ddot{\delta}_{x,k}} (-1)^x |k_n\rangle |1\rangle \right) \\ &= \frac{1}{\sqrt{2^{n+1}}} \sum_{k=0}^{2^{n+1}-1} (-1)^{\ddot{\delta}_{j,k}} |k_{n+1}\rangle. \end{aligned} \quad (16)$$

Then, let us go back to (10), and make pass the first N qubits through respective Hadamard gates. If we call $H^{\otimes N}$ this operation, the result is

$$\begin{aligned}
& H^{\otimes N} \left(\frac{1}{\sqrt{2^N}} \sum_{i=0}^{2^N-1} \alpha_i |i_N\rangle_1 \sum_{j=0}^{2^N-1} |(j \text{ XOR } i)_N j_N\rangle_{23} \right) \\
&= \frac{1}{\sqrt{2^N}} \sum_{i=0}^{2^N-1} \alpha_i \left[H^{\otimes N} |i_N\rangle_1 \sum_{j=0}^{2^N-1} |(j \text{ XOR } i)_N j_N\rangle_{23} \right] \\
&= \frac{1}{\sqrt{2^N}} \sum_{i=0}^{2^N-1} \alpha_i \left[\left(\frac{1}{\sqrt{2^N}} \sum_{k=0}^{2^N-1} (-1)^{\ddot{\delta}_{i,k}} |k_N\rangle_1 \right) \sum_{j=0}^{2^N-1} |(j \text{ XOR } i)_N j_N\rangle_{23} \right] \\
&= \frac{1}{2^N} \sum_{i,j,k=0}^{2^N-1} |k_N(j \text{ XOR } i)_N\rangle_{12} (-1)^{\ddot{\delta}_{i,k}} \alpha_i |j_N\rangle_3. \tag{17}
\end{aligned}$$

Observe that we have written between parenthesis in the third row in (17) the action of N Hadamard gates on the state of the first N qubits, i.e., $H^{\otimes N}|i_N\rangle_1$. This ends the proof (10) \Rightarrow (11).

B. Proof of (12).

In order to show our claim, we shall make use again of an argument based in the induction principle. Thus, we begin with the proof of (12) for the simplest case of $N = 2$, i.e., with the situation involving two qubits only. In this case, the general form of a two qubit state is given by

$$|\psi_2\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle, \tag{18}$$

where α_i , $i = 0, 1, 2, 3$ are complex numbers such that $\sum_{i=0}^3 |\alpha_i|^2 = 1$.

Thus, let us display (11) for $N = 2$:

$$\begin{aligned}
& \frac{1}{4} \sum_{i,j,k=0}^3 |k_2(j \text{ XOR } i)_2\rangle_{12} (-1)^{\ddot{\delta}_{i,k}} \alpha_i |j_2\rangle_3 \\
&= \frac{1}{4} [|0000\rangle_{12} (\alpha_0|00\rangle_3 + \alpha_1|01\rangle_3 + \alpha_2|10\rangle_3 + \alpha_3|11\rangle_3) \\
&\quad + |0001\rangle_{12} (\alpha_0|01\rangle_3 + \alpha_1|00\rangle_3 + \alpha_2|11\rangle_3 + \alpha_3|10\rangle_3) \\
&\quad + |0010\rangle_{12} (\alpha_0|10\rangle_3 + \alpha_1|11\rangle_3 + \alpha_2|00\rangle_3 + \alpha_3|01\rangle_3) \\
&\quad + |0011\rangle_{12} (\alpha_0|11\rangle_3 + \alpha_1|10\rangle_3 + \alpha_2|01\rangle_3 + \alpha_3|00\rangle_3) \\
&\quad + |0100\rangle_{12} (\alpha_0|00\rangle_3 - \alpha_1|01\rangle_3 + \alpha_2|10\rangle_3 - \alpha_3|11\rangle_3) \\
&\quad + |0101\rangle_{12} (\alpha_0|01\rangle_3 - \alpha_1|00\rangle_3 + \alpha_2|11\rangle_3 - \alpha_3|10\rangle_3) \\
&\quad + |0110\rangle_{12} (\alpha_0|10\rangle_3 - \alpha_1|11\rangle_3 + \alpha_2|00\rangle_3 - \alpha_3|01\rangle_3) \\
&\quad + |0111\rangle_{12} (\alpha_0|11\rangle_3 - \alpha_1|10\rangle_3 + \alpha_2|01\rangle_3 - \alpha_3|00\rangle_3) \\
&\quad + |1000\rangle_{12} (\alpha_0|00\rangle_3 + \alpha_1|01\rangle_3 - \alpha_2|10\rangle_3 - \alpha_3|11\rangle_3) \\
&\quad + |1001\rangle_{12} (\alpha_0|01\rangle_3 + \alpha_1|00\rangle_3 - \alpha_2|11\rangle_3 - \alpha_3|10\rangle_3) \\
&\quad + |1010\rangle_{12} (\alpha_0|10\rangle_3 + \alpha_1|11\rangle_3 - \alpha_2|00\rangle_3 - \alpha_3|01\rangle_3) \\
&\quad + |1011\rangle_{12} (\alpha_0|11\rangle_3 + \alpha_1|10\rangle_3 - \alpha_2|01\rangle_3 - \alpha_3|00\rangle_3) \\
&\quad + |1100\rangle_{12} (\alpha_0|00\rangle_3 - \alpha_1|01\rangle_3 - \alpha_2|10\rangle_3 + \alpha_3|11\rangle_3) \\
&\quad + |1101\rangle_{12} (\alpha_0|01\rangle_3 - \alpha_1|00\rangle_3 - \alpha_2|11\rangle_3 + \alpha_3|10\rangle_3) \\
&\quad + |1110\rangle_{12} (\alpha_0|10\rangle_3 - \alpha_1|11\rangle_3 - \alpha_2|00\rangle_3 + \alpha_3|01\rangle_3) \\
&\quad + |1111\rangle_{12} (\alpha_0|11\rangle_3 - \alpha_1|10\rangle_3 - \alpha_2|01\rangle_3 + \alpha_3|00\rangle_3)] \tag{19}
\end{aligned}$$

After an easy but rather cumbersome term by term analysis of the previous sum, we show that (19) is equal to

$$\frac{1}{4} \sum_{a,b,c,d=0}^1 |abcd\rangle_{12} (X^c \otimes X^d) (Z^a \otimes Z^b) |\psi_2\rangle_3, \tag{20}$$

where, as in the case $N = 1$, $X \equiv \sigma_x$ and $Z \equiv \sigma_z$, the Pauli matrices¹¹. Note that if M is any Pauli matrix, one has that $M^0 = I$, the identity matrix. Of course, $M^1 = M$.

Once we have proven our result for $N = 2$, the induction procedure assumes that the same is true for $N = 3, \dots, n$. Then, if we prove that the result is true for $N = n + 1$, it is

proven for any natural number N . Then, let us take the $n + 1$ qubit state given by

$$|\psi_{n+1}\rangle = \sum_{i=0}^{2^{n+1}-1} \alpha_i |i_{n+1}\rangle \quad (21)$$

To this end, we need to write formula (11) for $N = n + 1$ and to span it in a sum as follows:

$$\frac{1}{2^{n+1}} \sum_{i,j,k=0}^{2^{n+1}-1} |k_{n+1} (j \text{ XOR } i)_{n+1}\rangle_{12} (-1)^{\ddot{\delta}_{i,k}} \alpha_i |j_{n+1}\rangle_3$$

$$= \frac{1}{2} \left[\frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 0 (j \text{ XOR } i)_n 0\rangle_{12} (-1)^{\ddot{\delta}_{2i,2k}} \alpha_{2i} |j_n 0\rangle_3 \right. \quad (22)$$

$$+ \frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 0 (j \text{ XOR } i)_n 1\rangle_{12} (-1)^{\ddot{\delta}_{2i+1,2k}} \alpha_{2i+1} |j_n 0\rangle_3 \quad (23)$$

$$+ \frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 1 (j \text{ XOR } i)_n 0\rangle_{12} (-1)^{\ddot{\delta}_{2i,2k+1}} \alpha_{2i} |j_n 0\rangle_3 \quad (24)$$

$$+ \frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 1 (j \text{ XOR } i)_n 1\rangle_{12} (-1)^{\ddot{\delta}_{2i+1,2k+1}} \alpha_{2i+1} |j_n 0\rangle_3 \quad (25)$$

$$+ \frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 0 (j \text{ XOR } i)_n 1\rangle_{12} (-1)^{\ddot{\delta}_{2i,2k}} \alpha_{2i} |j_n 1\rangle_3 \quad (26)$$

$$+ \frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 0 (j \text{ XOR } i)_n 0\rangle_{12} (-1)^{\ddot{\delta}_{2i+1,2k}} \alpha_{2i+1} |j_n 1\rangle_3 \quad (27)$$

$$+ \frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 1 (j \text{ XOR } i)_n 1\rangle_{12} (-1)^{\ddot{\delta}_{2i,2k+1}} \alpha_{2i} |j_n 1\rangle_3 \quad (28)$$

$$+ \frac{1}{2^n} \sum_{i,j,k=0}^{2^n-1} |k_n 1 (j \text{ XOR } i)_n 0\rangle_{12} (-1)^{\ddot{\delta}_{2i+1,2k+1}} \alpha_{2i+1} |j_n 1\rangle_3 \quad (29)$$

Now, we use the induction hypothesis to each of the terms of the right hand side of the above relation. This gives the identity we write in the following long formula that should be understood in this sense: the row labelled as (22) is equal to the row labelled as (30),

(23) is equal to (31) and so on up to (29) equal to (37). In the next chain of formulas $3l$ is a subindex for the N first qubits of the third group and $3r$ labels the last qubit (the $3n + 3$ -th) of this group (now each group has $n + 1$ qubits by the induction hypothesis). Note that the forthcoming formula, although rather long, gives already the desired answer straightforwardly. Thus, the above relation equals to

$$\frac{1}{2} \left[\frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 0\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i} |i_n\rangle_{3l} |0\rangle_{3r} \right. \\ \left. + \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 1\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i+1} |i_n\rangle_{3l} |0\rangle_{3r} \right. \\ \left. + \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 0\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i} |i_n\rangle_{3l} |0\rangle_{3r} \right. \\ \left. + \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 1\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i+1} (-1) |i_n\rangle_{3l} |0\rangle_{3r} \right] \\ (30)$$

$$+ \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 0\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i} |i_n\rangle_{3l} |0\rangle_{3r} \\ (31)$$

$$+ \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 1\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i+1} (-1) |i_n\rangle_{3l} |0\rangle_{3r} \\ (32)$$

$$+ \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 1\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i+1} (-1) |i_n\rangle_{3l} |1\rangle_{3r} \\ (33)$$

$$+ \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 0\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i} |i_n\rangle_{3l} |1\rangle_{3r} \\ (34)$$

$$+ \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 0\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i+1} |i_n\rangle_{3l} |1\rangle_{3r} \\ (35)$$

$$+ \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 1\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i} |i_n\rangle_{3l} |1\rangle_{3r} \\ (36)$$

$$+ \frac{1}{2^n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 0\rangle_{12} \left(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k \right) \left(\bigotimes_{l=a_1}^{a_n} Z^l \right) \sum_{i=0}^{2^n-1} \alpha_{2i+1} (-1) |i_n\rangle_{3l} |1\rangle_{3r} \right]. \quad (37)$$

Let us analyze the above sum. It contains four kinds of terms:

- Two terms with $|0,0\rangle = |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 0\rangle$. These terms are (30) and (35). In fact, if we add up (30) and (35) we obtain a term of the form $2^{-n} \sum_{a_1, \dots, a_n=0}^1 |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 0\rangle_{12} (\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k) (\bigotimes_{l=a_1}^{a_n} Z^l)$ times the following sum

$$\sum_{i=0}^{2^n-1} \alpha_{2i+1} |i_n\rangle_{3l} |1\rangle_{3r} + \sum_{i=0}^{2^n-1} \alpha_{2i} |i_n\rangle_{3l} |0\rangle_{3r} = \sum_{i=0}^{2^{n+1}-1} \alpha_i |i_{n+1}\rangle_3 = |\psi_{n+1}\rangle. \quad (38)$$

In this case, Bob's state is obviously $(\bigotimes_{k=a_{n+1}}^{a_{2n}} X^k) (\bigotimes_{l=a_1}^{a_n} Z^l) |\psi_{n+1}\rangle$.

- Two terms with $|0,1\rangle = |a_1 \dots a_n 0 a_{n+1} \dots a_{2n} 1\rangle$, which are (31) and (34). In this case, the sum (38) is changed into

$$\sum_{i=0}^{2^n-1} \alpha_{2i} |i_n\rangle_{3l} |1\rangle_{3r} + \sum_{i=0}^{2^n-1} \alpha_{2i+1} |i_n\rangle_{3l} |0\rangle_{3r} = \sum_{i=0}^{2^{n+1}-1} \alpha_i X |i_{n+1}\rangle_3 = X |\psi_{n+1}\rangle. \quad (39)$$

Note that X applies to the last qubit only and the other n remain unchanged. Therefore, we should have rigorously written $I^{\otimes n} \otimes X$ to denote the tensor product n times the identity operator and one time X , but we have written X for simplicity.

- Two terms with $|1,0\rangle = |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 0\rangle$, which are (32) and (37). The sum gives here

$$Z |\psi_{n+1}\rangle. \quad (40)$$

As in the previous case, we have written Z instead of $I^{\otimes n} \otimes Z$ for simplicity.

- Two terms with $|1,1\rangle = |a_1 \dots a_n 1 a_{n+1} \dots a_{2n} 1\rangle$, which are (33) and (36). The sum is in this case

$$XZ |\psi_{n+1}\rangle. \quad (41)$$

Here, XZ replaces $(I^{\otimes n} \otimes X)(I^{\otimes n} \otimes Z)$.

Note the similarities with the $N = 1$ case. Finally, we conclude that the last long formula (30-37) is

$$\frac{1}{2^{n+1}} \sum_{a_1 \cdots a_{n+1}=0}^1 |a_1 \cdots a_{2n+2}\rangle_{12} \left(\bigotimes_{k=a_{n+2}}^{a_{2n+2}} X^k \right) \left(\bigotimes_{l=a_1}^{a_{n+1}} Z^l \right) |\psi_{n+1}\rangle_3. \quad (42)$$

Thus, we have obtained equation (12).

C. Description of the teleportation circuit.

The above mathematical presentation can be summarize in terms of what is call a *quantum circuit*. The circuit permits us a visualization of the computational process.

The input in the teleportation circuit (see Figure 1), is given by the N qubit state $|\psi_N\rangle$ and the $2N$ ancillary qubits all in the state $|0\rangle$. We readily see that

- i.) The first set of N ancillary qubits goes through Hadamard gates, which are here represented by the symbol \boxed{H} .
- ii.) CNOT operations involve two qubits and are represented by a dot \bullet in the control qubit (the qubit that determines the operation to be performed in the other qubit or target qubit), a wire that connects the dot with the symbol \oplus over the target qubit.
- iii.) Up to this point, all the operations are performed over the ancillary qubits. A dashed line separates these operations and all subsequent ones.
- iv.) After the dashed line, we make CNOT operations on the $2N$ first qubits as shown in the circuit.
- v.) The next operation is passing the first N quibits thorough respective Hadamard gates.
- vi.) A measurement process is carried out in the first $2N$ qubits. This is shown in the circuit by means of the symbol corresponding to a measurement apparatus.
- vii.) The horizontal dashed line separates between the Alice's qubits (above of the line) from the Bob's qubits (below of the line).
- viii.) Finally, the box in the lower right corner indicates the operation that Bob should perform in order to obtain the original N qubit state.

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⁷ In fact the vectors $|0\rangle$ and $|1\rangle$ represent the basis for any two level system. Among them the spin 1/2 is perhaps the most used. This is why, we use the language of spin up and spin down.

⁸ We recall that the Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad ; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

⁹ In fact, any vector in \mathbb{C}^2 has the matrix form given by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

where α and β are complex numbers. If we write as usual

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

we identify in a natural fashion the qubit $\alpha|0\rangle + \beta|1\rangle$ with the element of \mathbb{C}^2

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

If we have a system of N qubits, the Hilbert space that we need for such description is \mathbb{C}^{2^N} .

Each vector of \mathbb{C}^{2^N} (denoted as $|\psi_N\rangle$ or ψ_N) is a column matrix with 2^N complex entries. Each basis vector can be written in the form $|m_1 m_2 \dots m_N\rangle$, where m_i , $i = 1, 2, \dots, N$, are either 0 or 1 and is identified with a column matrix with all components equal to zero except one which gives 1.

¹⁰ Note that the CNOT operation is the quantum analog to the classical XOR operation.

¹¹ The tensor product of two 2×2 matrices is defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & b \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ c \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & d \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{pmatrix}.$$

The generalization to tensor products of $m \times m$ matrices is straightforward.

Figures

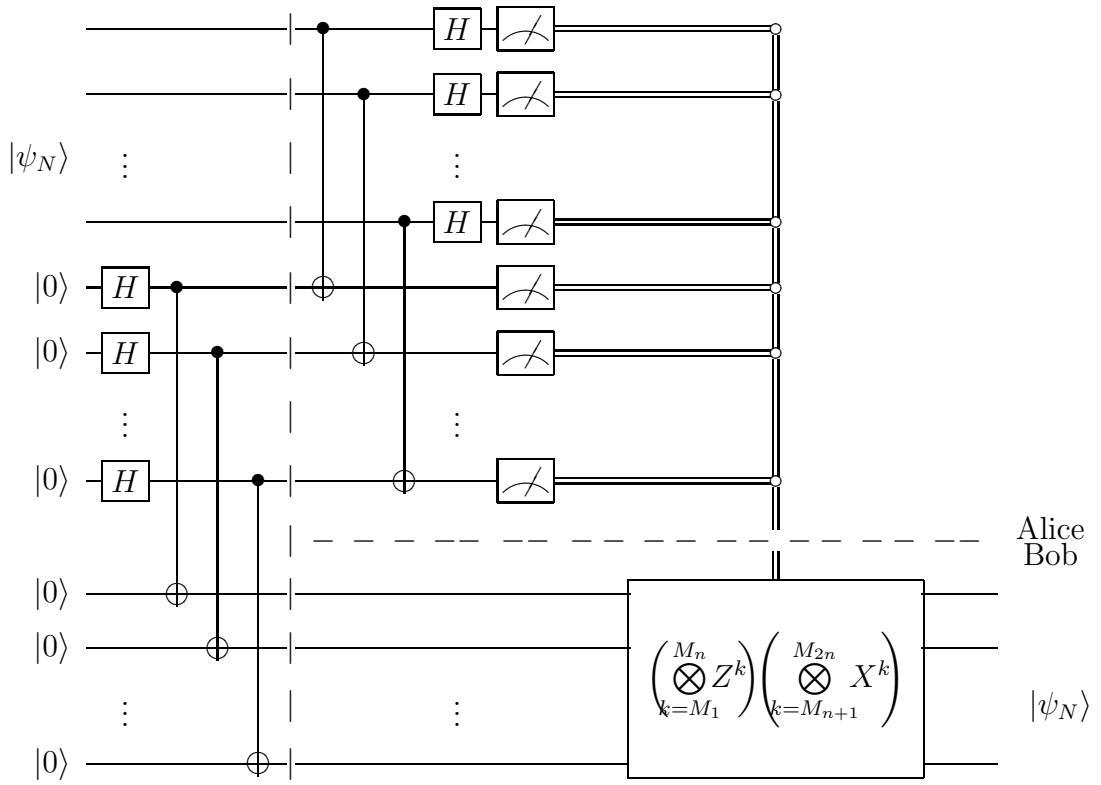


FIG. 1: N qubit state teleportation circuit